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THE GENERAL EQUATIONS OF ANALYTICAL DYNAMICS†

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It is shown that the generalized Poincaré and Chetayev equations, which represent the equations of motion of mechanical systems using a certain closed system of infinitesimal linear operators, are related to the fundamental equations of analytical dynamics. Equations are derived in quasi-coordinates for the case of redundant variables; it is shown that when an energy integral exists the operator $X_0 = \partial/\partial t$ satisfies the Chetayev cyclic-displacement conditions. Using the energy integral the order of the system of equations of motion is reduced, and generalized Jacobi–Whittaker equations are derived from the Chetayev equations. It is shown that the Poincaré–Chetayev equations are equivalent to a number of equations of motion of non-holonomic systems, in particular, the Maggi, Volterra, Kane, and so on, equations. On the basis of these, and also of other previously obtained results, the Poincaré and Chetayev equations in redundant variables, applicable both to holonomic and non-holonomic systems, can be regarded as general equations of classical dynamics, equivalent to the well-known fundamental forms of the equations of motion, a number of which follow as special cases from the Poincaré and Chetayev equations.

Poincaré's remarkable idea [1] and Chetayev's fine theory [2–4] on the application of Lie groups to represent the equations of motion of holonomic systems have been developed in a number of papers [5–20].

Thus, equations in variations for Poincaré's equations have been obtained and the existence of a principal invariant for the latter equations has been investigated in [5, 6]. The equations of motion of a system with an infinite number of degrees of freedom—a rigid body with a cavity containing a liquid, have been derived in the form of Poincaré and Chetayev equations [7]. One of the methods of constructing groups of possible displacements has been described in [8]. An extension and an application of the Chetayev cyclic displacements have been given and, in particular, an extension of Chaplygin's area theorem has been obtained, and some theorems of the interaction between the parts of a system have been established [9–11].

In a number of papers [12–14], Poincaré's equations were applied for the first time to non-holonomic systems, for which the system of operators of virtual displacements, as was shown, is not closed, whereas it is for holonomic systems. Poincaré's equations were derived by several methods for non-holonomic systems and it was shown that they are equivalent to many well-known equations, such as Appell's, Hamel's, Volterra's, Chaplygin's, Ferrers' and other equations.

In a number of papers [15–18], by means of a non-linear reversible replacement of the momenta, the Hamiltonian of the system was reduced to a form close to the Poincaré–Chetayev system. The consequences were a theorem on complete integrability, integrability on integral manifolds, and on classes of equivalence of Hamilton systems. A new method of obtaining particular solutions from familiar first integrals was proposed. For the case when the kinetic energy is independent of the coordinates, the conditions for a complete set of linear integrals to exist were established, and quadratures for these were obtained. By introducing redundant Poincaré parameters, equations of motion of non-holonomic systems were obtained for the case of stationary constraints, and expressions were derived for the reactions of the latter. Equations of the hydrodynamic type, and so on were obtained from the Poincaré–Chetayev equations.

It was proved in [19, 20] that the canonical Chetayev equations are the Hamilton equations in non-canonical variables. It was shown that the Lagrange and Hamilton systems of generalized equations in redundant coordinates, and also the equations in quasi-coordinates, are special cases of the Poincaré–Chetayev equations, the theory of which was thereby extended to these systems of equations. The equations of motion of non-holonomic systems were also derived in the form of Poincaré equations, which are outwardly different, but are equivalent to the equations derived in [12–14], and in the form of Chetayev equations.

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1. HOLONOMIC SYSTEMS

1.1. *Defining coordinates, parametrization of the constraints*

Consider a mechanical holonomic system with k degrees of freedom. The system position in space at any instant of time t is given by the governing coordinates [4] x_i ($i = 1, \ldots, n \geqslant k$), $\mathbf{r}_\upsilon = \mathbf{r}_\upsilon(t, x_1, \ldots, x_n)$ ($\upsilon = 1, \ldots, N$), where $\mathbf{r}_\nu$ is the radius vector of a mass point of mass m_υ. When $n = k$ the variables x_i are independent Lagrange coordinates, and when $n > k$ they are dependent, or redundant coordinates, subject to constraints, specified by the integrable system of differential equations

$$\eta_j \equiv a_{ji}(t, x_1, \ldots, x_n)\dot{x}_i + a_{j0}(t, x_1, \ldots, x_n) = 0, j = k+1, \ldots, n$$
$$\text{rank}\,(a_{ji}) = n-k, \dot{x}_i = dx_i/dt.$$

Everywhere summation is carried out over repeated subscripts.

The introduction of redundant coordinates is useful in some cases in order to simplify the expressions for the kinematic and dynamic quantities [21].

For symmetry and brevity we will conventionally put $t = x_0$

$$\eta_j \equiv a_{ji}(x)\dot{x}_i = 0, \; i = 0, 1, \ldots n, j = k+1, \ldots, n \tag{1.1}$$

where $x = (x_0, x_1, \ldots, x_n)$.

The sufficient conditions for Eqs (1.1) to be integrable, as is well known [22], have the form

$$\frac{\partial a_{jr}}{\partial x_s} = \frac{\partial a_{js}}{\partial x_r}, \; r, \; s = 0, 1, \ldots, n; \; j = k+1, \ldots, n \tag{1.2}$$

We will complete (1.1) by the arbitrarily chosen linear forms

$$\eta_s \equiv a_{si}(x)\dot{x}_i, \; s = 0, 1, \ldots, k, \; i = 0, 1, \ldots, n, \; \eta_0 = 1, a_{0i} = \delta_{0i} \tag{1.3}$$

which are linearly independent both of one another and in relation to the forms (1.1), so that det (a_{ij}) $\neq 0$ ($i, j = 0, 1, \ldots, n$), where δ_{ij} is the Kronecker delta.

Solving Eqs (1.1) and (1.3) for $\dot{x}_i$, we obtain the parametric representation of the constraints

$$\dot{x}_i = b_{is}(x)\eta_s, \; s = 0, 1, \ldots, k, \; i = 0, 1, \ldots, n, \; b_{0s} = \delta_{0s} \tag{1.4}$$

where $a_{si}b_{ir} = a_{ir}b_{si} = \delta_{sr}$.

1.2. *System of operators, Poincaré parameters*

Parametrization (1.4) enables us to construct a closed system of infinitesimal linear operators

$$X_s f \equiv b_{is}\frac{\partial f}{\partial x_i}, \; s = 0, 1, \ldots, k, \; f(x) \in C^2 \tag{1.5}$$

defining the virtual and real displacements of the system

$$\delta f = \omega_r X_r f, \quad r = 1, \ldots, k, \; df = \eta_s X_s f dt, \; s = 0, 1, \ldots, k \tag{1.6}$$

respectively, where $\omega_r \equiv a_{ri}(x)\delta x_i$ ($i = 1, \ldots, n$; $r = 1, \ldots, k$) and η_s are the parameters of the virtual and real displacements, introduced by Poincaré [1].

The system of operators (1.5) is a closed system in the sense that its commutator (the Poisson bracket) has the form

$$[X_r, X_s]f \equiv X_r X_s f - X_s X_r f = c_{rs}{}^m X_m f, \; m, r, s = 0, 1, \ldots, k \tag{1.7}$$

where the structural coefficients

$$c_{rs}^m = \left(\frac{\partial a_{mj}}{\partial x_i} - \frac{\partial a_{mi}}{\partial x_j}\right) b_{is} b_{jr} = a_{mj}\left(b_{ir}\frac{\partial b_{js}}{\partial x_i} - b_{is}\frac{\partial b_{ir}}{\partial x_i}\right), \; i, \; j = 0, 1, \ldots, n \tag{1.8}$$

where $c_{rs}^m = -c_{sr}^m, c_{0s}^0 = 0$ $(m, r, s = 0, 1, \ldots, k)$. The commutator is bilinear, skew-symmetric and satisfies the Jacobi identity; in turn it is first-order differential operator [4, 23].

By an appropriate choice of the auxiliary forms (1.3) we can reduce the closed system (1.5) to the Lie group, when all $c_{sr}^m = \text{const}$ [4]. That very case was considered by Poincaré and by Chetayev. However, in the general case of the closed system (1.5), the coefficients c_{sr}^m will, generally speaking, be variable, and this case is not excluded further from consideration.

Note that if the forms (1.3) are integrable, like forms (1.1), conditions (1.2) are also satisfied for $j = 1, \ldots, k$, and functions of the form $\pi_s = \pi_s(x)$ then exist which can serve as new *defining* coordinates, and besides $\eta_s = \dot{\pi}_s$ $(s = 1, \ldots, k)$, and also

$$\frac{\partial \pi_s}{\partial x_i} = \frac{\partial \eta_s}{\partial \dot{x}_i} = a_{si}, \quad \frac{\partial x_i}{\partial \pi_s} = \frac{\partial \dot{x}_i}{\partial \eta_s} = b_{is}$$

In this case, the system of operators (1.5) is an Abelian group of the functions

$$X_s f \equiv b_{is} \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial \pi_s}, \quad s = 0, 1, \ \ldots, \ k; \ i = 0, 1, \ \ldots, \ n$$

for which all the coefficients $c_{sr}^m = 0$ $(m, r, s = 0, 1, \ldots, k)$.

If the forms (1.3) are not integrable, the quantities $\pi_s(x)$ as functions of time and the coordinates do not exist, but the symbols π_s are reasonably introduced into consideration under the name of quasi-coordinates, using the conventional notation for the quasi-velocities $\eta_s = \dot{\pi}_s$ and the differentials of the quasi-coordinates $d\pi_s = \eta_s dt$, and also for the "partial derivatives with respect to the quasi-coordinates" and for the inverse relations [21]

$$\frac{\partial f}{\partial \pi_s} = b_{is} \frac{\partial f}{\partial x_i}, \quad \frac{\partial f}{\partial x_i} = a_{si} \frac{\partial f}{\partial \pi_s}, \quad i = 0, 1, \ \ldots, \ n; \ s = 0, 1, \ \ldots, \ k \tag{1.9}$$

By (1.9) the operators (1.5) in this case can be represented in the form

$$X_s f = \partial f / \partial \pi_s, \quad s = 0, 1, \ \ldots, \ k \tag{1.10}$$

with commutator (1.7), which takes the form

$$[X_r, \ X_s] f \equiv \frac{\partial^2 f}{\partial \pi_r \partial \pi_s} - \frac{\partial^2 f}{\partial \pi_s \partial \pi_r} = c_{rs}^m \frac{\partial f}{\partial \pi_m}$$

where, in general, $c_{rs}^m \neq 0$.

The parameters η_i and ω_i are linked by the following relations

$$d\omega_i/dt - \delta\eta_i = c_{sr}^i \eta_r \omega_s, \quad i, r, s = 0, 1, \ldots, k \tag{1.11}$$

initially established by Poincaré for the case when $c_{os}^i = 0$.

Note that expression (1.11) is equivalent to that for the external derivative of the form $\omega_i = a_j \delta x_j$. Expressions for the coefficients c_{rs}^i are more easily obtained from relations (1.11) than by using the general formulae (1.8) [21].

1.3. *Poincaré's equations*

Poincaré [1] and Chetayev [2, 3] used the Hamilton principle to derive the equations, while Chetayev [4] also used the d'Alembert–Lagrange principle in the traditional form

$$(m_\nu \ddot{\mathbf{r}}_\nu - \mathbf{F}_\nu) \cdot \delta \mathbf{r}_\nu = 0, \ \nu = 1, \ \ldots, \ N \tag{1.12}$$

We will use the d'Alembert–Lagrange principle in the defining coordinates [4]

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} - Q_i \right) \delta x_i = 0, \ i = 1, \ \ldots, \ n \tag{1.13}$$

when $L(t,x,\dot{x}) = T(t,x,\dot{x}) + U(t,x)$ is the Lagrange function, $T(t,x,\dot{x})$ is the kinetic energy, $U(t,x)$ is the force function, $x = (x_1,\dots,x_n)$, $\dot{x} = (\dot{x}_1,\dots,\dot{x}_n)$, $Q_i = \mathbf{F}_\nu^{*H}\cdot \partial \mathbf{r}_\nu/\partial x_i$ is the generalized non-potential force.

By (1.5) and (1.6) we have

$$\delta x_i = \omega_s X_s x_i = \omega_s b_{is},\ i = 1,\ \dots,\ n;\ s = 1,\ \dots,\ k$$

as a consequence of which, by virtue of the arbitrariness of ω_s, Maggi's equations of motion [24] follow from (1.13), namely

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} - Q_i\right)b_{is} = 0,\ \ s = 1,\ \dots,\ k \tag{1.14}$$

Using (1.4) we can express the Lagrange function in the form of the equality $L^*(t,x,\eta) = L(t,x,\dot{x})$, by differentiating which and using (1.3) we obtain the relations [25]

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L^*}{\partial \eta_m}a_{mi},\quad \frac{\partial L}{\partial x_i} = \frac{\partial L^*}{\partial x_i} + \frac{\partial L^*}{\partial \eta_m}\left(\frac{\partial a_{mj}}{\partial x_i}b_{jr}\eta_r + \frac{\partial a_{mo}}{\partial x_i}\right),\ \ i,\ j = 0, 1,\ \dots,\ n$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_m}\right)a_{mi} + \frac{\partial L^*}{\partial \eta_m}\left(\frac{\partial a_{mi}}{\partial x_j}b_{jr}\eta_r + \frac{\partial a_{mi}}{\partial t}\right),\ \ r = 0, 1,\ \dots,\ k$$

substitution of which into (1.14) leads to Poincaré's equations

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s} = c_{rs}^m\frac{\partial L^*}{\partial \eta_m}\eta_r + c_{0s}^m\frac{\partial L^*}{\partial \eta_m} + X_s L^* + Q_s^*,\ \ m,\ r,\ s = 1,\ \dots,\ k \tag{1.15}$$

where $Q_s^* = Q_i b_{is}$. The structural coefficients

$$c_{rs}^m = \left(\frac{\partial a_{mj}}{\partial x_i} - \frac{\partial a_{mi}}{\partial x_j}\right)b_{is}b_{jr},\ \ i,\ j = 1,\ \dots,\ n$$

$$c_{0s}^m = \left[\left(\frac{\partial a_{mj}}{\partial x_i} - \frac{\partial a_{mi}}{\partial x_j}\right)b_{j0} + \frac{\partial a_{m0}}{\partial x_i} - \frac{\partial a_{mi}}{\partial t}\right]b_{is} \tag{1.8*}$$

elaborate expressions (1.8) for the explicit selection of $t = x_0$.

Equations (1.15) together with Eqs (1.14) form a compatible system of $k + n$ first-order differential equations of motion, each with the same number of unknowns $\eta_1,\dots,\eta_k, x_1,\dots,x_n$. It is noteworthy that Eqs (1.15) in redundant coordinates contain no reaction forces of the constraints (1.1) and have the same outward appearance in both independent coordinates ($n = k$) and dependent coordinates ($n > k$).

Poincaré's equations (1.15) contain, as special cases, the equations, first given by Poincaré [1] for the case when $n = k$, $X_0 = \partial/\partial t$, $c_{0i}^s = 0$, $Q_i^* = 0$ ($i = s = 1,\dots,k$), the Lagrange equations of the second kind when $n = k$, $\eta_s = \dot{x}_s$, $X_s = \partial/\partial x_s$, when all $c_{rs}^m = 0$, and the generalized Lagrange equations [19] in redundant coordinates

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{x}_s} - \frac{\partial L^*}{\partial x_s} - b_{js}\frac{\partial L^*}{\partial x_j} = Q_s^*,\ \ s = 1,\ \dots,\ k;\ \ j = k+1,\ \dots,\ n$$

when the constraints are specified in the form $\dot{x}_j = b_{js}\dot{x}_s$ ($s = 0, 1,\dots,k; j = k+1,\dots,n$), and also the generalized Boltzmann [26]–Hamel [27] equations in quasi-coordinates

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{\pi}_s} = c_{rs}^m\dot{\pi}_r\frac{\partial L^*}{\partial \dot{\pi}_m} + c_{0s}^m\frac{\partial L^*}{\partial \dot{\pi}_m} + \frac{\partial L^*}{\partial \pi_s} + Q_s^*,\ \ m,\ r,\ s = 1,\ \dots,\ k \tag{1.16}$$

where $L^* = L^*(t, x_1, \ldots, x_n; \pi_1, \ldots, \dot{\pi}_1, \ldots, \dot{\pi}_k)$.

Equations (1.16) are a unique form of the Boltzmann equations in the case of dependent coordinates (after eliminating indefinite coefficients), and the Boltzmann–Hamel equations in the case of independent coordinates. In particular, Euler's equations of motion of a rigid body around a fixed point follow from Eqs (1.16).

Equations (1.15) and (1.4), under certain conditions, admit of certain first integrals.

If

$$X_0 = \partial/\partial t, \quad c_{0i}^s = 0,\ X_0 L^* = 0, \quad Q_s^* = 0,\ i, s = 1, \ldots, k \tag{1.17}$$

the following energy integral exists

$$\frac{\partial L^*}{\partial \eta_i}\eta_i - L^* = h = \text{const}, \quad i = 1, \ldots, k$$

which, in the general case when $L^* = L_2^* + L_1^* + L_0^*$, where L_s^* are homogeneous forms of the variables η_i of degree s ($s = 0, 1, 2$), take the form $L_2^* - L_0^* = h$.

Note that conditions (1.17) are satisfied if all $a_{s0} = b_{i0} = 0$ in Eqs (1.3) and (1.4), while the coefficients a_{si} and b_{is} are explicitly independent of time, like the Lagrange function L^* $(x_1, \ldots, x_n, \eta_1, \ldots, \eta_k)$.

For the case $Q_i^* = 0$ ($i = 1, \ldots, k$) Chetayev [3, 4] introduced the idea of cyclic displacements X_α ($\alpha = l + 1, \ldots, k$), which satisfy the conditions

$$1)\ [X_\alpha, X_s] = 0,\ s = 0, 1, \ldots, k, \quad 2)\ X_\alpha L^* = 0 \tag{1.18}$$

When $Q_s^* = 0$ and conditions (1.18) hold, Eqs (1.15) have the first integrals

$$\partial L^* / \partial \eta_\alpha = \beta_\alpha = \text{const}, \quad \alpha = l + 1, \ldots, k \tag{1.19}$$

Using integrals (1.19) Chetayev constructed a generalized Routh function and showed that Poincaré's questions take the form of generalized Routh equations [3, 4] for non-cyclic displacements of X_s ($s = 1, \ldots, l$).

Comparing conditions (1.17) and (1.18) we see that the operator $X_0 = \partial/\partial t$ satisfies conditions (1.18), i.e. is an operator of cyclic displacements, to which the energy integral corresponds.

Indeed, if we represent Poincaré's equations in parametric form, when the time $t = x_0$ and the coordinates x_i ($i = 1, \ldots, n$) are considered as variables x_α ($\alpha = 0, 1, \ldots, n$) that are independent of one another subject to differential constraints (1.1) and specified by continuous differentiable functions of a certain parameter τ, $x_\alpha = x_\alpha(\tau)$, the energy integral of the Poincaré parametric equations with Lagrangian [28] $L^*(x_i, \eta_s)\, x_0'$, $x_0' = dt/d\tau$ will correspond to the variable x_0.

Using the energy integral we can reduce the order of the system of equations by determining from the integral the variable

$$x_0' = t' = \varphi(x_i, \eta_s, h)$$

and constructing the Routh function

$$R(x_i, \eta_s, h) = L^* x_0' - \left(L^* - \frac{\partial L^*}{\partial \eta_s}\eta_s\right)x_0' = \frac{\partial L^*}{\partial \eta_s}\eta_s x_0' \tag{1.20}$$

on the right-hand side of which the variable x_0' is replaced by the function $\varphi(x_i, \eta_s, h)$. The parametric Routh equations with $\tau = t$ take the form of Poincaré's equations (1.15) in which all $c_{0s}^m = 0$, $Q_s^* = 0$.

If we take one of the quantities π_s, say π_1, as the parameter τ, we can obtain from the energy integral

$$\eta_1 = \dot{\pi}_1 = 1/t' = \psi(x_i, \eta_r', h),\ r = 2, \ldots, k$$

where $\eta_r' = d\pi_r/d\pi_1 = \eta_r/\eta_1$, $t' = dt/d\pi_1 = 1/\eta_1$ and, substituting into (1.20), we obtain the new Routh function

$$R'(x_i,\ \eta'_r,\ h) = \frac{\partial L^*}{\partial \eta_s}\frac{\eta_s}{\eta_1} \tag{1.21}$$

It is easy to show [25], that the following equalities hold

$$\frac{\partial R'}{\partial \eta'_r} = \frac{\partial L^*}{\partial \eta_r},\ \frac{\partial R'}{\partial x_i} = \frac{1}{\eta_1}\frac{\partial L^*}{\partial x_i},\ i = 1,\ \dots,\ n;\ r = 2,\ \dots,\ k$$

substituting which into (1.16) for $s = 2, \dots, k$ and all $c_{0s}^m = Q_s^* = 0$, we obtain the generalized Jacobi [29]–Whittaker [25] equations in quasi-coordinates

$$\frac{d}{d\pi_1}\frac{\partial R^*}{\partial \eta'_s} = c_{rs}^m \eta'_r \frac{\partial R'}{\partial \eta'_m} + \frac{\partial R}{\partial \pi_s},\ s = 2,\ \dots,\ k \tag{1.22}$$

If we put $\eta_1 = \dot{x}_1$ in (1.3) (then [26] $c_{rs}^1 = 0$), Eqs (1.22) take the form

$$\frac{d}{dx_1}\frac{\partial R'}{\partial \eta'_s} = c_{rs}^m \eta'_r \frac{\partial R'}{\partial \eta'_m} + \frac{\partial R'}{\partial \pi'_s},\ m,\ r,\ s = 2,\ \dots,\ k \tag{1.23}$$

As pointed out in Section 1.2, when Eqs (1.3) are integrable the variables π_s can serve as new *defining* coordinates. Then $c_{rs}^m = 0$ and Eqs (1.23) take the form of the Jacobi–Whittaker equations.

Equations (1.23) need to be investigated in the general case together with the constraint equations (1.4), written in the following form

$$x' = b_{is}\eta'_s,\ i = 2, \dots, n;\ s = 2, \dots, k \tag{1.24}$$

Equations (1.23) and (1.24) can be regarded as the equations of motion of a new dynamical system with $k - 1$ degrees of freedom, for which R' is the kinetic potential, η_r are the parameters of the real displacements, while x_1 plays the part of time as an independent variable. The dependence of x_1 on the time t is established by quadrature [25].

1.4. *The canonical Chetayev equations*

Chetayev [3, 4] converted Poincaré's equations to canonical form by introducing, instead of η_s and $L^*(t, x, \eta)$, new variables y_s and function $H^*(t, x, y)$, defined by the equations

$$y_s = \frac{\partial L^*}{\partial \eta_s},\ s = 1,\ \dots,\ k,\ H^*(t, x,\ y) = y_s\eta_s - L^*(t, x,\ \eta) \tag{1.25}$$

which yield the following equations

$$X_s H^* = -X_s L^*,\ \eta_s = \frac{\partial H^*}{\partial y_s},\ s = 1,\ \dots,\ k \tag{1.26}$$

Transformation (1.25) is a Legendre transformation, if we take into account the fact that $\| \partial^2 L^*/\partial\eta_r\partial\eta_s \| \neq 0$, $(r, s = 1, \dots, k)$. Since

$$y_s = \frac{\partial L^*}{\partial \eta_s} = \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \eta_s} = p_i b_{is},\ \eta_s = a_{sj}\dot{x}_j,\quad p_i = \frac{\partial L}{\partial \dot{x}_i}$$

it is obvious that the following equality holds

$$H^*(t, x, y) = p_i b_{is} a_{sj}\dot{x}_j - L(t, x, \dot{x}) = H(t, x, p)$$

(the formula $a_{sj}b_{is} = \delta_i$ is taken into account).

Substituting (1.25) and (1.26) into Poincaré's equations (1.15) yields the canonical Chetayev equations

$$\frac{dy_s}{dt} = c_{rs}^m \frac{\partial H^*}{\partial y_r} y_m + c_{os}^m y_m - X_s H^* + Q_s^*,\ \eta_s = \frac{\partial H^*}{\partial y_s},\quad m,\ r,\ s = 1,\ \dots,\ k \tag{1.27}$$

These equations need to be investigated in the general case together with Eqs (1.4), by means of which the second group of equations (1.27) can be given another form [4]

$$\frac{dx_i}{dt} = X_o x_i + \frac{\partial H^*}{\partial y_s} X_s x_i, \quad i = 1, \ldots, n \tag{1.28}$$

Note that, like Eqs (1.15), the first group of equations (1.27) can be derived directly from Eqs (1.14), rewritten in the form

$$\left(\frac{dp_i}{dt} + \frac{\partial H}{\partial x_i} - Q_i\right) b_{is} = 0, \quad s = 1, \ldots, k$$

(equations of the form (1.26) are taken into account).

Chetayev's equations (1.27) contain the following as special cases.

1. The canonical Hamilton equations, when the variables x_i are independent Lagrange coordinates ($n = k$), while the group (1.5) is reduced to commutation transformations, where the Lagrange generalized velocities $\eta_i = \dot{x}_i$ are taken as the parameters of the real displacements, the variables x_i, p_i will be canonical coordinates, while $H(t, x, p)$ is the classical Hamilton function.

2. The generalized Hamilton equations in redundant coordinates [19]

$$\frac{dy_s}{dt} = -\frac{\partial H^*}{\partial x_s} - b_{js}\frac{\partial H^*}{\partial x_j} + Q_s^*, \quad \frac{dx_s}{dt} = \frac{\partial H^*}{\partial y_s}, \quad s = 1, \ldots, k; \; j = k+1, \ldots, n$$

3. The canonical Boltzmann–Hamel equations in quasi-coordinates [19]

$$\frac{dy_s}{dt} = c_{rs}^m \frac{\partial H^*}{\partial y_r} y_m + c_{os}^m y_m - \frac{\partial H^*}{\partial \pi_s} + Q_s^*, \quad \frac{d\pi_s}{dt} = \frac{\partial H^*}{\partial y_s}$$

For Eqs (1.27) with $Q_s = 0$ the generalized Jacobi and Poisson theorems hold [2–4] (the latter under certain additional conditions).

When $X_0 = \partial/\partial t$, $c_{0i}^s = 0$, $X_0 H^* = 0$, $Q_s^* = 0$ $(i, s = 1, \ldots, k)$, equivalent to conditions (1.17), Eqs (1.27) have the energy integral

$$H^*(x_1, \ldots, x_n, y_1, \ldots, y_k) = \text{const}$$

equivalent to the energy integral of Eqs (1.15).

When the cyclic displacements X_α $(\alpha = l+1, \ldots, k)$ exist under conditions (1.18), Eqs (1.27) with $Q_s^* = 0$ will allow of the integrals

$$y_\alpha = \beta_\alpha = \text{const}, \; \alpha = l+1, \ldots, k$$

similar to integrals (1.19) of Eqs (1.15). For non-cyclic displacements X_i Eqs (1.27) take the form of the equations

$$\frac{dy_i}{dt} = c_{ri}^s \frac{\partial H^*}{\partial y_r} y_s + c_{ri}^\alpha \frac{\partial H^*}{\partial y_r} \beta_\alpha + c_{0i}^\alpha \beta_\alpha - X_i H^* + Q_i^*, \qquad \eta_i = \frac{\partial H^*}{\partial y_i} \tag{1.29}$$

$$i, r, s = 1, \ldots, l; \; \alpha = l+1, \ldots, k$$

equivalent to the generalized Routh equations [3, 4], where $H^* = H^*(t, x_1, \ldots, x_n, y_1, \ldots, y_l, \beta_{l+1}, \ldots, \beta_k)$. After integrating Eqs (1.29) the variables η_α are defined by the equations $\eta_\alpha = \partial H^*/\partial\beta_\alpha$ $(\alpha = l+1, \ldots, k)$.

Using the energy integral we can reduce Chetayev's equations by two orders. In fact, suppose the integral $H^*(x_i, y_s) + h = 0$ is solvable with respect to the variable y_1, so that

$$y_1 + K(x_1, \ldots, x_n, y_2, \ldots, y_k, h) = 0$$

Consider the Legendre transformation for Eqs (1.22) and (1.23)

$$y'_r = \partial R'/\partial\eta'_r,\quad K(x,\ y'_r,\ h) = y'_r\eta'_r - R'(x,\ \eta'_s,\ h),\quad r = 2,\ \ldots,\ k$$

which yields the equations $X_rK = -X_rR'$, $\eta'_r = \partial K/\partial y'_r$.

Taking these equations into account, Eqs (1.22) can be written in the form of the generalized Whittaker equations [25]

$$\frac{dy'_s}{d\pi_1} = c^m_{rs}\frac{\partial K}{\partial y'_r}y'_m - \frac{\partial K}{\partial \pi_s},\quad m,r,\ s = 2,\ \ldots,\ k,\quad \frac{dt}{d\pi_1} = \frac{\partial K}{\partial h},\quad \frac{dh}{d\pi_1} = 0 \tag{1.30}$$

and also Eqs (1.23)

$$\frac{dy'_s}{dx_1} = c^m_{rs}\frac{\partial K}{\partial y'_r}y'_m - \frac{\partial K}{\partial \pi_s},\quad m,\ r,\ s = 2,\ \ldots,\ k,\quad \frac{dt}{dx_1} = \frac{\partial K}{\partial h},\quad \frac{dh}{dx_1} = 0 \tag{1.31}$$

The last pairs of equations (1.30) and (1.31) can be separated from the remaining equations since the first $2(k-1)$ equations do not contain t, while h = const. Hence, the first $2(k-1)$ equations of (1.30) or (1.31) can be regarded as the equations of motion of the reduced system with $k-1$ degrees of freedom [25].

2. NON-HOLONOMIC SYSTEMS

2.1. *Poincaré's and Chetayev's equations for non-holonomic systems*

Poincaré's equations, like the Boltzmann–Hamel equations in quasi-coordinates, are used to describe both holonomic and non-holonomic systems. This problem has already been investigated in [12–15], as well as in [19], where Chetayev's equations were also considered in this sense.

We must, however, emphasize, that the system of operators of virtual displacements for non-holonomic systems is not closed [14, 15], as a result of which one must use operators of the corresponding holonomic system, obtained from the non-holonomic system considered by mentally discarding non-integrable constraints.

When considering non-holonomic systems, and also the Boltzmann–Hamel equations, the case when there were no integrable constraints was considered in [19]. Here, we will consider the general case of a non-holonomic system in redundant coordinates, subject to integrable constraints

$$\eta_j \equiv a_{ji}(x)\dot{x}_i = 0,\ \mathrm{rank}\ (a_{ij}) = n-k,\ i = 0,1,\ \ldots,n;\ j = k+1,\ \ldots,n \tag{2.1}$$

and non-integrable constraints

$$\eta_\alpha \equiv a_{\alpha i}(x)\dot{x}_i = 0,\ \mathrm{rank}\ (a_{\alpha i}) = \mathrm{k}-1,\ \alpha = l+1,\ \ldots,k \tag{2.2}$$

We will arbitrarily choose linear differential forms

$$\eta_s \equiv a_{si}(x)\dot{x}_i,\ \mathrm{s} = 0,1,\ \ldots,l;\ a_{0i} = \delta_{oi} \tag{2.3}$$

independent of one another, and also with forms (2.1) and (2.2), $\det(a_{ij}) \neq 0$ $(i,j = 0, 1, \ldots, n)$. In particular, we can take the generalized velocities $\dot{x}_s$ as the quantities η_s, $(s = 1, \ldots, l)$.

Solution of forms (2.1)–(2.3) leads to the equations

$$\dot{x}_i = b_{is}(x)\eta_s,\ i = 0,1,\ \ldots,n,\ s = 0,1,\ \ldots,l;\ b_{i0} = \delta_{i0} \tag{2.4}$$

For the corresponding holonomic system, obtained by mentally discarding the non-integrable constraints (2.2) considered, i.e. assuming $\eta_\alpha \neq 0$ $(a = l+1, \ldots, k)$, instead of (2.4) we obtain the equations

$$\dot{x}_i = b_{is}(x)\eta_s,\ i = 0,1,\ \ldots,n;\ s = 0,1,\ \ldots,k$$

and we construct the closed system of operators (1.5).

Since the parameters of the virtual displacements $\omega_\alpha = 0$ when the constraints (2.2) are present, from the d'Alembert–Lagrange principle (1.13) we derive the equations of motion of a non-holonomic system of the form (1.15)

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s} = c_{rs}^m \eta_r \frac{\partial L}{\partial \eta_m} + c_{os}^m \frac{\partial L^*}{\partial \eta_m} + X_s L^* + Q_s^*, \quad r, \ s = 1, \ \ldots, \ l; \ m = 1, \ \ldots, k \tag{2.5}$$

the number of which is less than the number of equations (1.15) by $k-1$. The structural coefficients c_{rs}, as previously, are given by formulae (1.18) in which, however, the subscripts $r, s = 0, 1, \ldots, l$. Note that Eqs (2.5) have the same outward appearance as Eqs (5.3) [19] in independent coordinates. We must add the constraint equations (2.4) to Eqs (2.5), as a result of which we obtain a compatible system of $l + n$ equations of motion with the same number of unknowns $x_1, \ldots, x_n, \eta_1, \ldots, \eta_l$.

Note that the function $L^*(t, x, \eta)$, which occurs in Eqs (2.5), constructed for the corresponding holonomic system, may, in general, depend on all the parameters η_r ($r = 1, \ldots, k$), as a consequence of which the constraint equations $\eta_\alpha = 0$ ($\alpha = l+1, \ldots, k$) need only be taken into account after setting up Eqs (2.5) [27, 21, 30].

Note that Eqs (2.5), when $Q_s^* = 0$, are equivalent to Eqs (3.14) [12] and (1.13) [13], but are outwardly somewhat simpler due to the choice of the parameters η_α, which vanish by virtue of the equations of the non-integrable constraints (2.2).

For the cases when the generalized velocities $\dot{x}_s = \eta_s$ ($s = 0, 1, \ldots, l$) are taken as the parameters η_s (2.3), i.e. when $a_{si} = \delta_{si}$ ($i = 1, \ldots, n$), all the structural coefficients [26] $c_{rs}^m = 0$ for $m \leqslant l$, and Eqs (2.5) take the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{x}_s} = c_{rs}^m \frac{\partial L^*}{\partial \eta_m}\dot{x}_r + c_{os}^m \frac{\partial L^*}{\partial \eta_m} + X_s L^* + Q_s^* \quad r, \ s = 1, \ \ldots, \ l, \quad m = l+1, \ \ldots, \ k \tag{2.6}$$

where $L^* = L^*(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_l, \eta_{l+1}, \ldots, \eta_k)$.

If in the functions $L^*(t, x, \eta)$ in Eqs (2.5) we replace the kinetic energy $T^*(t, x, \eta_1, \ldots, \eta_k)$ of the corresponding holonomic system by the kinetic energy $\Theta(t, x, \eta_1, \ldots, \eta_l)$ of the non-holonomic system with constraints (2.2), Eqs (2.5) take the form of Eqs (5.5) [19]

$$\begin{gathered}\frac{d}{dt}\frac{\partial \theta}{\partial \eta_s} = (c_{rs}^m \eta_r + c_{os}^m)\frac{\partial \theta}{\partial \eta_m} + (c_{rs}^p \eta_r + c_{os}^p)\left(\frac{\partial T^*}{\partial \eta_p}\right) + X_s(\theta + U) + Q_s^* \\ m, \ r, \ s = 1, \ \ldots, \ l; \ p = l+1, \ \ldots, \ k\end{gathered} \tag{2.7}$$

where $(\partial T^*/\partial \eta_p)$ denote the expressions $\partial T^*/\partial \eta_p$ with $\eta_s = 0$ ($p, s = 1+1, \ldots, k$).

Using the Legendre transformation (1.25) of Eq. (2.5), the motions of the non-holonomic system can be written in the form of Chetayev's canonical equations

$$\begin{gathered}\frac{dy_s}{dt} = c_{rs}^m \frac{\partial H^*}{\partial y_r} y_m + c_{os}^m y_m - X_s H^* + Q_s^*, \ \eta_s = \frac{\partial H^*}{\partial y_s} \\ r, \ s = 1, \ \ldots, \ l; \ m = 1, \ \ldots, \ k\end{gathered} \tag{2.8}$$

to which we must add the constraint equations (2.2) and relations (2.4), rewritten in the form

$$\frac{\partial H^*}{\partial y_\alpha} = 0, \ \alpha = l+1, \ \ldots, \ k; \ \frac{dx_i}{dt} = b_{ij}\frac{\partial H^*}{\partial y_j}, \ i = 1, \ \ldots, \ n; \ j = 0, 1, \ \ldots, \ l \tag{2.9}$$

Equations (2.8) and (2.9) form a complete system of $n + k + 1$ equations with the same number of unknowns $x_1, \ldots, x_n, y_1, \ldots, y_k, \eta_1, \ldots, \eta_l$. Equations (2.8) and (2.9) with $n = k$ take the form of Eqs (7.17)–(7.19) of [30].

The canonical equations of motion of non-holonomic systems, equivalent to Eqs (2.7), have the form

$$\begin{gathered}\frac{dy_s}{dt} = \left(c_{rs}^m \frac{\partial H^*}{\partial y_r} + c_{os}^m\right) y_m + \left(c_{rs}^p \frac{\partial H^*}{\partial y_r} + c_{os}^p\right)\left(\frac{\partial T^*}{\partial \eta_p}\right) - X_s H^* + Q_s \\ \eta_s = \partial H^* / \partial y_s, \ m, \ r, \ s = 1, \ldots, l; \ p = l+1, \ \ldots, \ k\end{gathered} \tag{2.10}$$

where $H^* = y_s \eta_s - \theta - U$.

2.2. The equivalence of Poincaré's and Chetayev's equations to different forms of the equations of motion

Previously [12–14] it was shown by direct calculations that Poincaré's equations of motion of non-holonomic systems are equivalent to Chaplygin's, Appell's, Hamel's, Volterra's, Ferrers' and certain other equations. The equivalence of the equations in quasi-coordinates to Appell's equations, and also to Chaplygin's equations was proved in [30] by deriving these groups of equations from the d'Alembert–Lagrange principle. The Voronets equations were derived from Poincaré's equations (5.6) in [19].

We will show that Poincaré's equations are equivalent to certain other forms of equations of motion of non-holonomic systems.

In Section 1.3 Poincaré's equations were derived from Maggi's equations [24] (1.14). Similarly, Eqs (2.5) are equivalent to Eqs (1.14) when (2.2) is taken into account.

As Maggi showed [24], both Appell's equations and Volterra's equations follow from his equations.

Maggi considered a mechanical system with coordinates x_i ($i = 1, \ldots, n$) subject to m linear constraints, which can be both holonomic and non-holonomic, and explicitly dependent or independent of time. By solving the constraint equations for $\dot{x}_i$, he presented them in the form (2.4), referring to the quantities η_s (in his notation—e_s) as the characteristics of the motion of the system considered, where $b_{is} = \partial\dot{x}_i/\partial\eta_s = \partial\ddot{x}_i/\partial\dot{\eta}_s$ ($s = 1, \ldots, l = n - m$). Proceeding to the derivation of Volterra's equations, Maggi converted his equations of the form (1.14) (in which the kinetic energy T occurs instead of L, while Q denotes all the active forces applied to the system) to the form

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_r} = \frac{db_{ir}}{dt}\frac{\partial T}{\partial \dot{x}_i} + b_{ir}\frac{\partial T}{\partial x_i} + P_r, \quad r = 1, \ldots, l; \quad P_r = Q_i b_{ir} \tag{2.11}$$

Volterra [31] considered a system with N point masses, the velocities of which in a Cartesian system of coordinates are related to the characteristics of the motion by relations of the form (2.4)

$$\dot{x}_i = b_{is}\eta_s, \; i = 1, \ldots, 3N, \; s = 1, \ldots, l$$

where $b_{is} = b_{is}(x_1, \ldots, x_{3N})$. Here Maggi's equations (2.11) take the form of Volterra's equations [31]

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_r} = c_{rs}^{(k)}\eta_k\eta_r + P_r, \quad k, \; r, \; s = 1, \ldots, l \tag{2.12}$$

$$\left(c_{rs}^{(k)} = m_i b_{ik}\frac{\partial b_{ir}}{\partial x_j} b_{js}; \quad m_i = m_{i+1} = m_{i+2}; \quad i, \; j = 1, \ldots, 3N \right)$$

where $T(x_1, \ldots, x_{3N}, \eta_1, \ldots, \eta_l)$ is the kinetic energy.

Without giving Maggi's derivation of Appell's equations from Eqs (1.14) here we note that they are simpler to derive directly from Eq. (1.12). Differentiating Eqs (2.4) with respect to time we have $\ddot{x}_i = b_{is}(x)\dot{\eta}_s + \ldots$, where the dots denote terms not containing $\dot{\eta}_s$. Hence we find that $\partial\ddot{x}_i/\partial\dot{\eta}_s = b_{is}$, as a result of which, from (1.12), we obtain Appell's equations

$$\partial S / \partial\dot{\eta}_s = \Pi_s, \quad s = 1, \ldots, l \tag{2.13}$$

where $S = m_\nu\ddot{\mathbf{r}}_\nu^2/2$ is the energy of the accelerations and $\Pi_s = \mathbf{F}_\nu \cdot \mathbf{b}_{s\nu}$ is the generalized force referred to the quasi-coordinate π_s [30].

We will show, finally, that Kane's equations are equivalent to Poincaré's equations. By (1.6) $\delta\mathbf{r}_\nu = \omega_s X_s \mathbf{r}_\nu$ ($\nu = 1, \ldots, N, s = 1, \ldots, l$). Substituting these expressions into (1.12) we obtain the equations of motion in the form

$$m_\nu\ddot{\mathbf{r}}_\nu \cdot X_s\mathbf{r}_\nu = \mathbf{F}_\nu \cdot X_s\mathbf{r}_\nu, \quad s = 1, \ldots, l \tag{2.14}$$

For a system with Lagrangian coordinates q_i subject to non-integrable constraints

$$\dot{q}_j = b_{js}(t, q)\dot{q}_s + b_j(t, q), j = l + 1, \ldots, n; s = 1, \ldots, l$$

and operators (1.5) of the form

$$X_s f = \frac{\partial f}{\partial q_s} + b_{js}\frac{\partial f}{\partial q_j}$$

Eqs (2.14) are identical with Kane's equations [32, Eqs (19)]

$$K_{q_s} + K'_{q_s} = 0, \quad s = 1, \ldots, l$$

which, consequently, are equivalent to Eqs (2.5).

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